Adiabatic Quantum Transport

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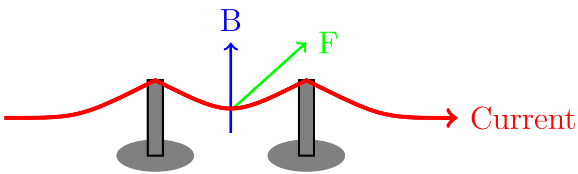
1 The Hall effect

1.1 Maxwell, Hall and Rowland, (1879)

E. H. Hall (American J. Mathematics, 1879)

SOMETIME during the last University year, while I was reading Maxwell's Electricity and Magnetism in connection with Professor Rowland's lectures, my attention was particularly attracted by the following passage in Vol. II, p. 144: "It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it.



1.2 The classical Hall effect

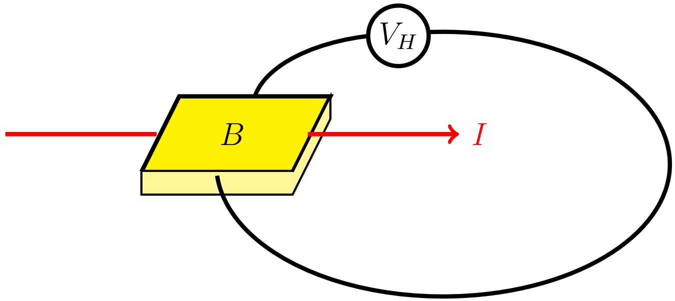


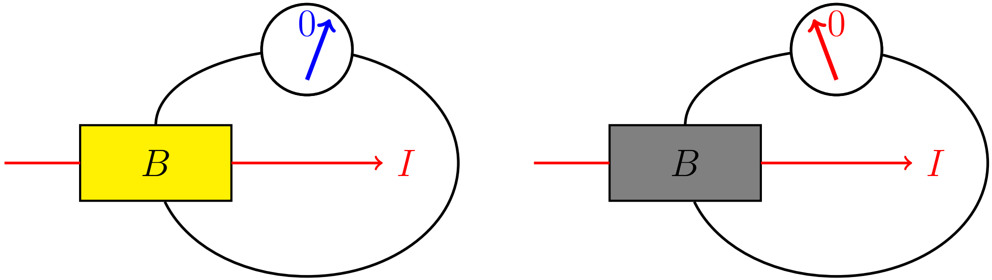
Figure 1: Hall experiment

Driving current	B	Response
.03	11,000	.000,000,0013

- $V_H \propto \frac{B}{n_\square} I, \quad n_\square$ =area density of charges
- $B \gg 1 \quad \wedge \quad n_\square \ll 1 \Longrightarrow$ thin (semi) conductors.

1.3 Peierles, Bloch and Heisenberg (1929)

- Puzzle: $\operatorname{sgn} V \Longleftrightarrow$ electrons/holes



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- Energy bands $\Longrightarrow E(\mathbf{k})$ periodic $\Longrightarrow$ electrons/holes

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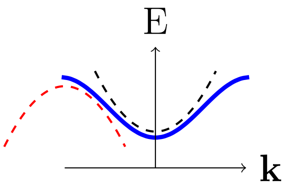
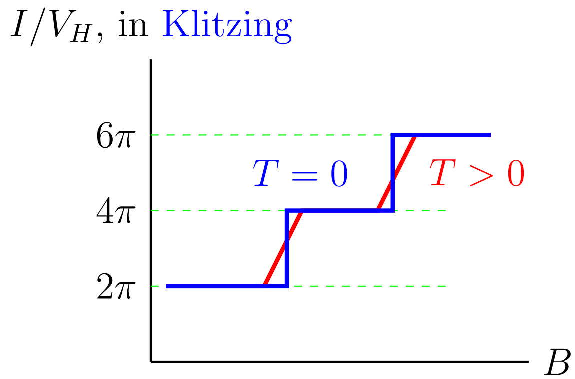
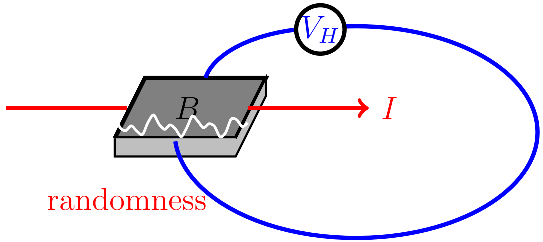


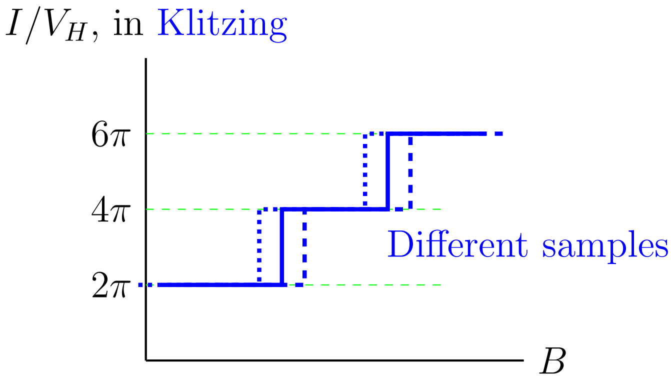
1.4 von Klitzing and Laughlin (1980)

- Hall conductance I/V_H
- Quantum unit of conductance:
- Klitzing$=\frac{\mathrm{e}^2}{\hbar} = \underbrace{\frac{2\pi}{25,812.807,455,5}}_{12 \text{ digits}\Omega^{-1}}$

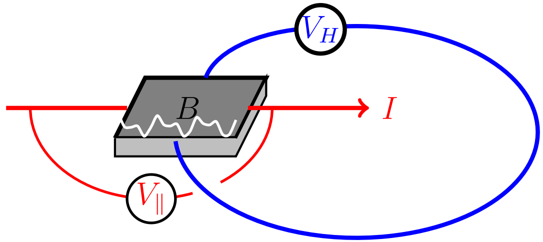


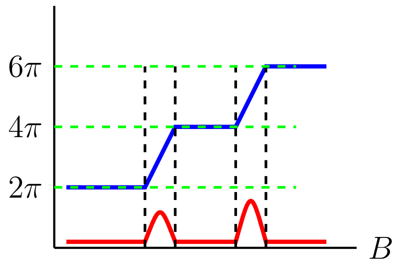
1.5 Generic precision





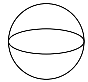
1.6 Quantization hand in hand with no dissipation





1.7 Metrology: Natural vs precision

- Antiquity: T(Astronomy)
- French revolution:L(Earth), M(Water)



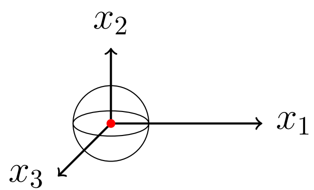
- Industrial revolution: M & L artifacts in Sevres
- Quantum revolution: T(atomic physics), L(c,T)
-

	Planck	Units	SI	accuracy
c	L/T	1	2.997,924,58	exact
$\hbar$	L^2M/T	1	$1.054,571,800 \times 10^{-34}$	10 digits
G	$L^3/(MT^2)$	1	6.67408×10^{11}	6 digits

- Getting rid of the last artifact: Kg
  - Avogadro: Count to 10^{23}.
  - Newton: Watt balance calibration uses QHE.

1.8 What Hall conductance counts?

- Laughlin: Electrons/holes
- TKNN:
  - Chern numbers
  - A feature of quantum states
  - Quantum analog of Degree of a map
  - $\mathbf{x} : S^2 \mapsto \mathbb{R}^3/0$



$$Degree = \frac{1}{4\pi} \int_{S^2} \frac{d\mathbf{x} \times d\mathbf{x} \cdot \mathbf{x}}{\mathbf{x}^3} \in \mathbb{Z}$$

1.9 What Hall conductance counts? (cont)

- Bellissard
  - Fredholm index
  - A feature of quantum Hilbert space
  - Infinite dimensional analog of

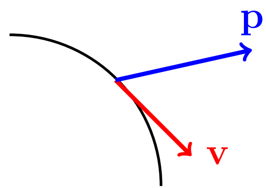
$$\underbrace{Index\, F}_{n\times m} = m - n \in \mathbb{Z}$$

1.10 What's right and wrong with Laughlin?

- Right: Physical conditions
  - Macroscopy
  - $T << B$
  - 2D
- Problematic: Conceptual structure
  - Charge operator: $Q = \sum a_j^* a_j$
  - Spect $(Q) \in \mathbb{Z}$
  - $\underbrace{\langle\psi| Q |\psi\rangle \in \mathbb{R}}_{\text{Expectations smooths}}$
- Chern numbers: Analog of Dirac charge quantization.

2 Free particles in magnetic field

- $L = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \underbrace{\mathbf{A}\cdot\mathbf{v}}$ (m=1)
- Forces by gauge invariance of Euler-Lagrange
- $\underbrace{\mathbf{p} = \partial_v L = \mathbf{v} + \mathbf{A}}_{\text{Gauge dependent}}$
- $H = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2$



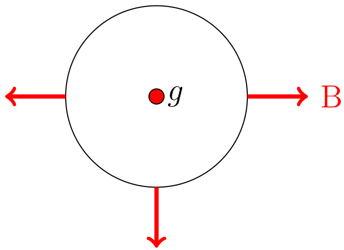
2.1 Gauge covariance

- Gauge transformation: $UU^* = \mathbb{1}, \quad U\mathbf{x} = \mathbf{x}U$
- $\underbrace{v_\mu = -i\partial_\mu - A_\mu}_{\text{covariant space-time derivatives}}$
- $\{v_\mu,\ |\psi\rangle\} \equiv \{Uv_\mu U^*,\ U|\psi\rangle\}$
- $Uv_\mu U^* = v_\mu + iU^*\partial_\mu U = \underbrace{-i\partial_\mu - A'_\mu}_{\text{redefine A}}$
- Covariant Evolution: $\big(\underbrace{v_0}_{v_t} + H(\mathbf{v}, \mathbf{x})\big)\,|\psi\rangle = 0$
- Indeed: $U\big(v_0 + H(\mathbf{v}, \mathbf{x})\big)U^*U|\psi\rangle = 0$
- Functional calculus: $UH(v_j, x_k)U^* = H(Uv_jU^*, x_k)$

2.2 Dirac monopoles: Quantization enforced by gauge invariance

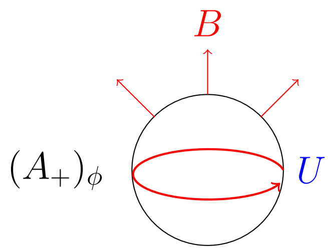
- $g=$ Charge of magnetic monopole
- e = electric charge
- Dirac : $2g\left(\frac{\mathrm{e}}{\hbar}\right) \in \mathbb{Z} \quad (\hbar = 1)$

$|\langle x|\psi\rangle|^2 \neq 0$



2.3 $A_\pm$ for monopole

- Monopole: $\mathbf{B} = g \frac{\mathbf{x}}{|\mathbf{x}|^3}$
- By Gauss $\nabla \times \mathbf{B} = 4\pi g \delta(\mathbf{x})$
- North hemisphere $\underbrace{\oint A_+ = \int B_+}_{Stokes}$
- $\underbrace{(A_+)_\phi \sin\theta}_{\text{line integral}} = \underbrace{g\, r(1 - \cos\theta)}_{flux}$
- Singular at antartica $\theta = \pi$
- Similarly A_- regular at south hemisphere



2.4 Dirac quantization

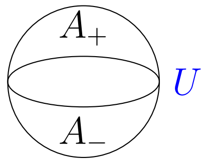
- $\underbrace{e(A_+ - A_-) = iU^* dU}_{\text{gauge invariance}}, \quad \underbrace{\langle x|\psi_-\rangle = U(x)\langle x|\psi_+\rangle}_{|U(x)|=1}$

- Putting the two hemispheres together again

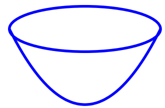
$$\underbrace{4\pi g}_{Gauss}\ e = \underbrace{e\int_{equator} (A_+ - A_-)}_{stokes}$$

$$= \underbrace{i\int_{equator} U^* dU}_{\text{Gauge invariance}}$$

$$\in 2\pi \underbrace{\mathbb{Z}}_{\text{winding of U}}$$



2.5 Quantum particles in magnetic field



- Schrödinger :

$$H_s = \frac{1}{2} \underbrace{\mathbf{v} \cdot \mathbf{v}}_{Euclidean}, \quad \mathbf{v} = -i\nabla - \mathbf{A},$$

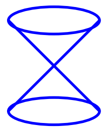
- Pauli: Spin 1/2

$$H_p = \frac{1}{2}(\mathbf{v} \cdot \sigma)^2 = \underbrace{H_s}_{Zeeman} \otimes \underbrace{\mathbb{1}}_{spin} - \frac{1}{2} \underbrace{\mathbb{1}}_{orbit} \otimes \underbrace{\sigma \cdot B}_{Pauli}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Dirac-Weyl: 2D

$$H_d = \mathbf{v} \cdot \sigma = \begin{pmatrix} 0 & v_1 + iv_2 \\ v_1 - iv_2 & 0 \end{pmatrix}, \quad v_f = 1$$



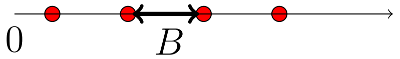
2.6 Homogeneous B – 2D: Landau levels

- Velocities canonically conjugate.

$$\underbrace{[v_1, v_2] = [-i\partial_1 - A_1, -i\partial_2 - A_2] = i(\partial_1 A_2 - \partial_2 A_1) = iB}_{[x,p]=i\hbar}$$

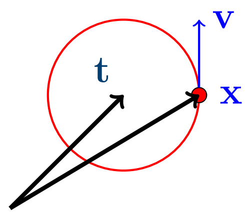
- H_s, H_p and H_d^2 are all harmonic oscillators:

$$\text{Spec }(H_s) = B\left(n + \frac{1}{2}\right),$$
$$\text{Spec }(H_p) = B\, n,$$
$$\text{Spec }(H_d) = \pm\sqrt{(B/2)\, n}, \quad n = 0, 1, \ldots, \infty$$



2.7 Constant of motion

- $B = homogeneous \Longrightarrow$ Translation/Gauge
- Nöther: translation symmetry $\Longrightarrow \underbrace{\text{conserved "momenta"}}_{\text{generators of translations}}$
- Translation/Gauge $\Longrightarrow$ generate non-commuting translations
- $[H_{s,p,d}, \mathbf{t}] = 0, \quad \mathbf{t} = \underbrace{-i\nabla - \mathbf{A}(\mathbf{x})}_{\mathbf{v}} - \mathbf{x} \times B$
- $t_{a,b}$ canonical: $\underbrace{[t_1, t_2] = i\,B}_{[x,p]=i\hbar}$
- $\underbrace{[t_j, v_k] = 0}_{commuting}$
- $(\mathbf{x}, \mathbf{p}) \leftrightarrow (\mathbf{v}, \mathbf{t})$

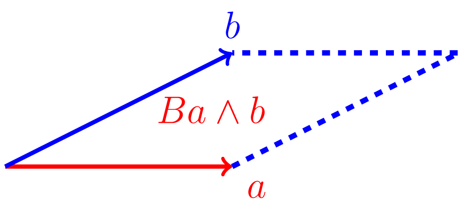


2.8 Magnetic translations

- $\underbrace{T_{\mathbf{a}} = e^{-i\mathbf{t}\cdot\mathbf{a}}}_{\text{Unitary}}, \quad \mathbf{a} \in \mathbb{R}^2$

- Weyl group: $T_{\mathbf{a}} T_{\mathbf{b}} = \underbrace{e^{iB\,\mathbf{a}\wedge\mathbf{b}}}_{\text{magnetic flux}} \; T_{\mathbf{b}} T_{\mathbf{a}}$

- $(T_{\mathbf{a}}\psi)(\mathbf{x}) = \underbrace{e^{i\Lambda(\mathbf{a},\mathbf{x})}}_{\text{re-Gauge}} \psi(\mathbf{x}-\mathbf{a}), \quad \Lambda(\mathbf{a},\mathbf{x}) = -\mathbf{a}\cdot\mathbf{A} + B\mathbf{x}\wedge\mathbf{a}$

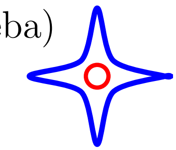


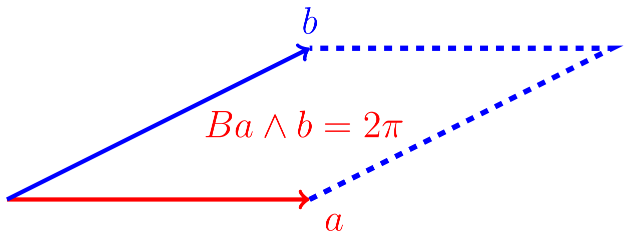
2.9 Qflux

- Qflux: $Ba \wedge b \underbrace{\left(\frac{\mathrm{e}}{\hbar}\right)}_{=1} = 2\pi$;

- Earth magnetic flux through micro-organism (amoeba)

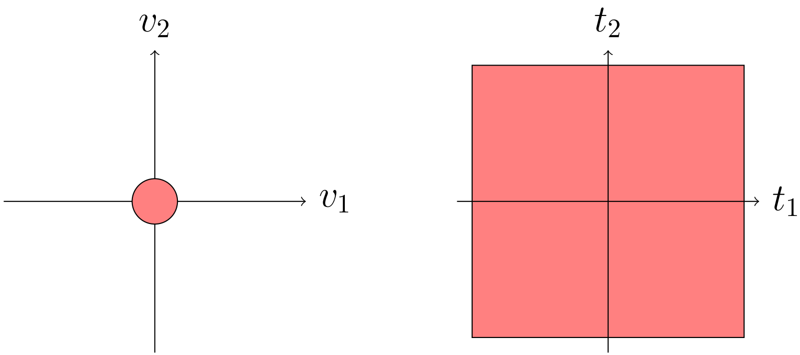
- Magnetic translations enclosing qflux commute





2.10 Life in a Landau

- The plane is non commutative
- **t** plane like phase space: $[t_1, t_2] = iB$



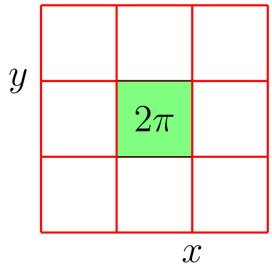
2.11 Weyl calculus

- $\underbrace{f(x,y)}_{c-numbers}$: classical observable in the plane
- $\underbrace{\hat{f}(k_x,k_y)}_{c-numbers}$ Fourier transform
- Weyl quantization:

$$\underbrace{F}_{quantization} = \frac{1}{2\pi}\int \hat{f}(k_x,k_y)\underbrace{T_{\mathbf{k}/B}}_{unitary}\, dk_x dk_y$$

- Quantum-classical correspondence:

$$TrF = \frac{B}{2\pi}\int f(x,y)dxdy$$



- State per Qflux

2.12 Charge particle on a torus

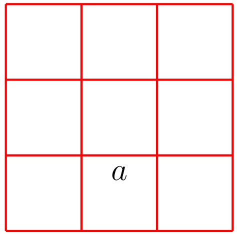


Figure 2: Torus: $\mathbb{R}^2/\mathbb{Z}^2$

- Standard Periodic boundary conditions

$$\underbrace{\psi(\mathbf{x}-a) \neq e^{ik_a}\psi(\mathbf{x})}_{\text{inconsistent with gauge invariance}}$$

- "Gauge periodic boundary" conditions,

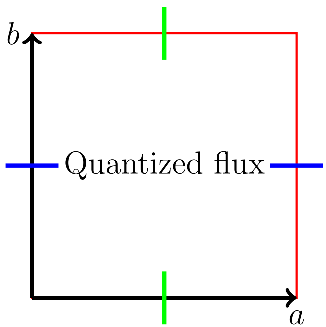
$$T_a\left|\psi\right\rangle = e^{ik_a}\left|\psi\right\rangle, \quad \psi(\mathbf{x}-a) = e^{ik_a}\underbrace{e^{iG(\mathbf{x})\cdot a}}_{re-gauge}\psi(\mathbf{x})$$

2.13 Dirac quantization (again)

- Boundary conditions: $T_a T_b |\psi\rangle = \underbrace{e^{i(k_a+k_b)}}_{\text{global phase}} |\psi\rangle = T_b T_a |\psi\rangle$

- Weyl: $T_a T_b = e^{iBa\wedge b}\, T_b T_a$

- $\underbrace{B\,\mathbf{a}\wedge\mathbf{b}}_{\text{flux}} \in 2\pi\mathbb{Z}$



2.14 Breaking of translation invariance

- Phase of $\langle x|\psi\rangle$ winds n times around the torus.
- The density $|\psi(\mathbf{x})|^2$ has n zeros (can't be uniform!)
- What broke translation symmetry?

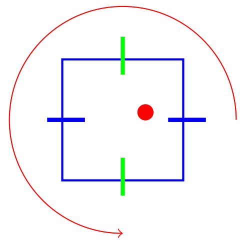
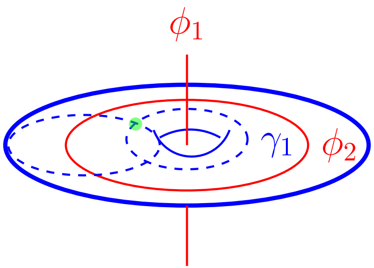
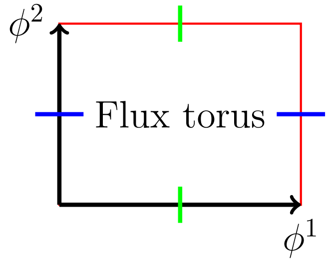


Figure 3: Twisted bc imply nodes in density

2.15 Aharonov-Bohm fluxes–Jacobi torus

- B does not determine A,
- $\mathbf{A}$ determines (B, ϕ_1, ϕ_2)
- Aharonov-Bohm fluxes: $\phi_j = \int_{\gamma_j} \mathbf{A}$
- The assignment ϕ_j (choice γ_j) breaks symmetry.





- Translation invariance recovered by the torus of fluxes

2.16 Time reversal

- $T\mathbf{x} = \mathbf{x}T, \quad T\mathbf{p} = -\mathbf{p}T \quad \wedge \quad [p_j, x_k] = -i\delta_{jk} \Longrightarrow \underbrace{Ti = -iT}_{anti-unitary}$
- $\underbrace{T\boldsymbol{\sigma} = -\boldsymbol{\sigma}T}_{spin\ 1/2}$
- $$T = \begin{cases} * & \text{Integer spin} \\ \sigma_y * & \text{Half Integer spin} \end{cases} \qquad T^2 = \begin{cases} \mathbb{1} & \text{Integer spin} \\ -\mathbb{1} & \text{Half Integer spin} \end{cases}$$
- $\underbrace{\langle\psi|\varphi\rangle = \overline{\langle T\varphi|T\psi\rangle}}_{anti-unitary}$
- Kramer: $\langle\psi|T\psi\rangle = \overline{\langle T^2\psi|T\psi\rangle} = \pm\langle\psi|T\psi\rangle$

Exercise 1. *Show that T has no eigenvectors*

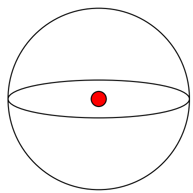
2.17 QM on manifolds

- H_s

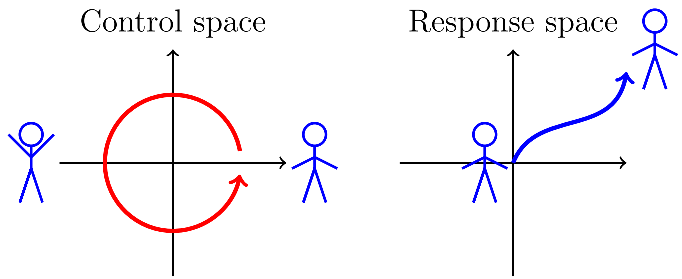
$$2\,\langle\psi|\,H_s\,|\psi\rangle = \langle\psi|\,\underbrace{\mathbf{v}\cdot\mathbf{v}}_{metric}\,|\psi\rangle = \int d\mathbf{x}\,\sqrt{g}\,\overline{(v_j\psi)}(\mathbf{x})\,g^{jk}(v_k\psi)(\mathbf{x})$$

- $H_{p,d}$

$$\underbrace{\mathbf{v}\cdot\sigma}_{\text{spin connection}} = v_j e_a^j \sigma^a, \quad e_j^a e_k^b \delta_{ab} = g^{jk}$$

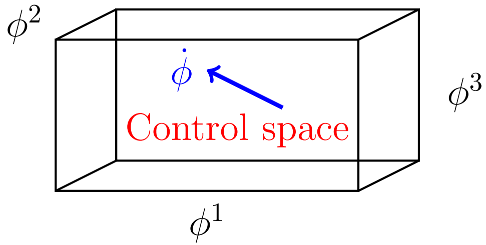


3 Controls and Response

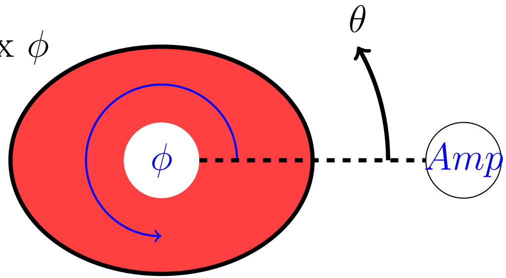


3.1 Driving and response

- Controls: ϕ
- Hamiltonian $H(\phi)$
- Driving: $\underbrace{\dot{\phi} = \frac{d\phi}{dt}}_{vector}$
- Response: $\underbrace{d(\text{virtual work})}_{1-form} = -dH = -\frac{\partial H}{\partial \phi}\,d\phi$



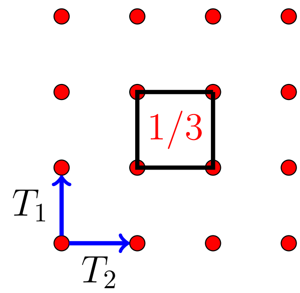
3.2 Electromotive forces and loop currents



- Control: Aharonov-Bohm flux ϕ
- $\mathbf{A} = \frac{\phi}{2\pi} d\theta = \phi\, \delta(\theta) d\theta$
- Drive: emf $\dot{\phi} = -\oint E_j dx^j$
- Minimal coupling: $H(\mathbf{p} - \mathbf{A})$
- Heisenberg equation: $\mathbf{v} = i[H, \mathbf{x}] = \partial_p H$
- Response : Loop current

$$
\begin{aligned}
-dH &= -(\partial_\phi H)\, d\phi \\
&= \frac{1}{2}(\mathbf{v} \cdot \partial_\phi \mathbf{A} + \partial_\phi \mathbf{A} \cdot \mathbf{v})\, d\phi \\
&= \frac{1}{2}(v_\theta \delta(\theta) + \delta(\theta) v_\theta)\, d\phi
\end{aligned}
$$

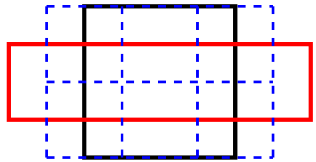
3.3 Electric fields and currents



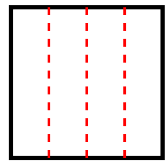
- Bloch Hamiltonians: Hofstadter model
- $H(\mathbf{k}) = e^{ik_1}\mathbf{T}_1 + e^{ik_2}\mathbf{T}_2 + h.c.$
- $\mathbf{T}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$
- $B = 1/3 \Leftrightarrow \omega = e^{2\pi i/3}$
- Control: Brillouin zone $\quad \mathbf{k} \in T^2$
- Driving: External electric field: $\dot{\mathbf{k}} = \mathbf{E}$
- Response: $-dH = -(\partial_k H)dk = \underbrace{-i(T_j - T_j^*)}_{velocity} dk^j$

3.4 Drive: Strain rate, response: Stress

- Control space: Metric tensor g
- Linear deformations L
- Active :$\mathbf{x}' = L^{-1}\mathbf{x}$; Passive $g' = L^t\, gL$
- $(d\ell)^2 = (d{\mathbf{x}'}^t)g'(d\mathbf{x}') = (d\mathbf{x}^t)\, g\, (d\mathbf{x})$
- Driving: strain rate $= \dot{g}$
- Response: Stress $dH(g) = -(\partial_g H)\, dg$



Active



Passive

3.5 Transport coefficients

- Conductances: $\underbrace{I_a}_{\text{a loop current}} = \underbrace{\sigma_{ab}}_{Conductance} \underbrace{\dot{\phi}^b}_{emf}$

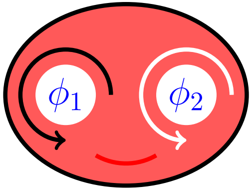
- Viscosity: $\underbrace{\sigma_{\alpha\beta}}_{stress} = \underbrace{\eta_{\alpha\beta\gamma\delta}}_{viscosity} \underbrace{\dot{g}^{\gamma\delta}}_{\text{strain rate}}$

- Non-Dissipative $I_a\dot{\phi}^a = 0, \quad \sigma_{\alpha\beta}\dot{g}^{\alpha\beta} = 0$

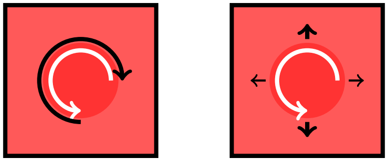
- Non-Dissipative: Response tensors anti-symmetric.

3.6 Odd (non-dissipative) response

- White arrow: Driving.
- Black arrow response.



Charge transport



Viscosity

3.7 Odd and Isotropic in 2D

- Conductance is econd rank prop to Levi-Civita:
$$\underbrace{\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{Levi-Civita}$$

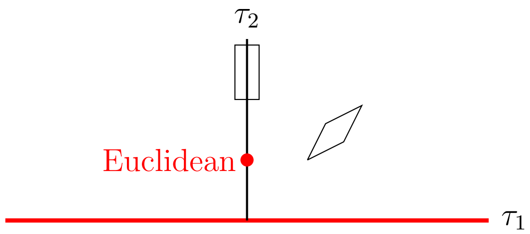
- Odd viscosity is fourth rank prop to:

$$\sigma_1 \otimes \sigma_3 - \sigma_3 \otimes \sigma_1$$

Exercise 2. *Why? What happend in 3D?*

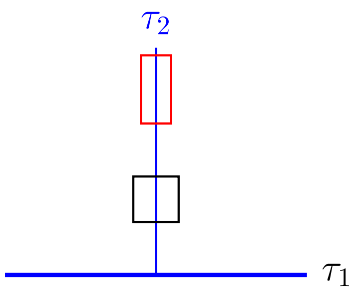
3.8 The space of flat metrics with $\det g = 1$

- $g(\tau) = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \quad \tau = \tau_1 + i\,\tau_2 \quad \tau_2 > 0$
- $\det g = \frac{|\tau|^2 - \tau_1^2}{tau_2^2} = 1$
- $g(\tau)$ flat
  - curvature=0
  - No holonomy of parallel transport.

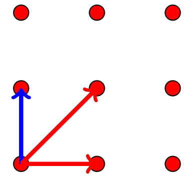


3.9 The metric on the space of flat metrics

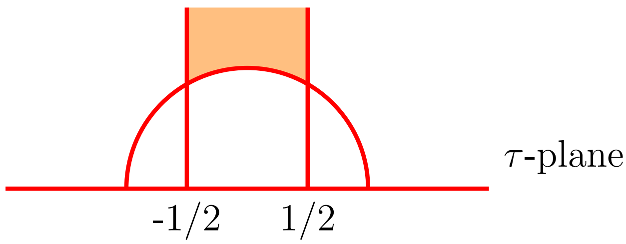
- $SL(2,\mathbb{R}) : L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det L = 1, \quad a,b,c,d \in \mathbb{R}$
- $g = L^t \cancel{SO(2)}^t \mathbb{1} \cancel{SO(2)} L$
- $\tau \leftrightarrow SL(2,\mathbb{R})/SO(2)$
- $g'(\tau') = g(\tau), \quad g' = L^t\, gL, \quad \tau' = \frac{a\tau+b}{c\tau+d}$
- The Haar measure is on the upper half-plane is then $\frac{d\tau_1 d\tau_2}{\tau_2^2}$



3.10 The fundamental domain of $SL(2,\mathbb{Z})$



- Torus: $\mathbb{R}^2/\mathbb{Z}^2$
- $L \in SL(2,\mathbb{Z}) : \mathbb{Z}^2 \mapsto \mathbb{Z}^2$
- $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$:
- Fundamental domain
  - 2D
  - Finite area
  - Two conic points
  - 1 cusp

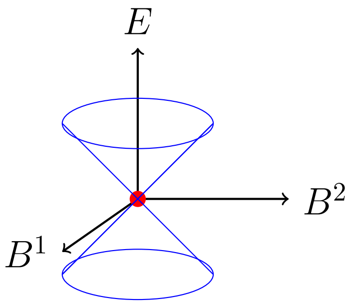


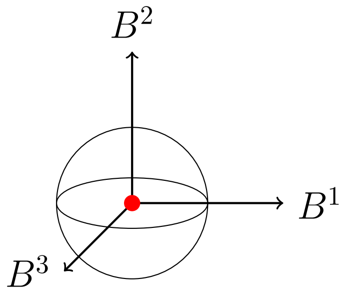
3.11 Gapped Hamiltonians

- Gap condition: Endows control space with interesting topology

Example 3.1.

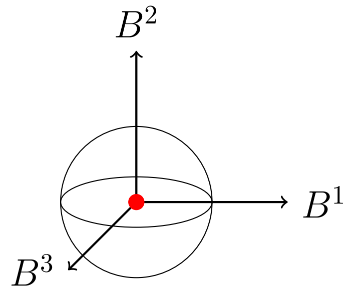
$$H(\mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\sigma}, \quad \mathbf{B} \in \underbrace{\mathbb{R}^3/0}_{S^2}$$





3.12 Wigner von Neuman

- Simple=Non-degenerate
- dim(Hermitian $n \times n$) $= n^2$
- dim(Hermitian, 2-fold degenerate) $= n^2 - 3$



3.13 Proof

- Simple: $H = U(n)\not{U}(1)^{\otimes n} \underbrace{D}_{diagonal} \not{U}^*(1)^{\otimes n} U^*(n)$

- $H \sim \underbrace{\text{cone}(\lambda_1 < \lambda_2 \cdots < \lambda_n)}_{eigenvalues} \otimes \underbrace{U(n)/U(1)^n}_{frame}$

- 1-eigenvalue crossing

$$H = U(n)\not{U}(1)^{\otimes n-2}\not{U}(2) \underbrace{D}_{diagonal} \not{U}(2)\not{U}^*(1)^{\otimes n-2} U^*(n)$$

$$H \sim \text{cone}(\lambda_1 = \lambda_2 \cdots < \lambda_n) \otimes U(n)/ \underbrace{U(2)}_{\mathbb{1}_2 V = V \mathbb{1}_2} \otimes U(1)^{n-2}$$

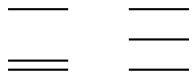
- Co-dimension: $\underbrace{1}_{cone} + \underbrace{4}_{U(2)} - \underbrace{2}_{U(1)} = 3$

Exercise 3. *co-dim (real symmetric)* $= 2$

3.14 Homotopy of simple matrices

- (simple hermitian $n \times n$) $\underbrace{\sim}_{homotopic}$ $U(n)/U(1)^n$

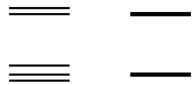
- Deform $(\lambda_1 < \lambda_2 \ldots \lambda_n)$ to $(1, 2, \ldots, n)$.



- (simple 2×2) $\sim S^2$

$$\underbrace{\pi_{0,1}(S^2) = 0, \quad \pi_{2,3}(S^2) = \mathbb{Z}, \quad \pi_{5,6}(S^2) = \mathbb{Z}_2}_{\text{rich topology}}$$

3.15 Homotopy of gapped matrices

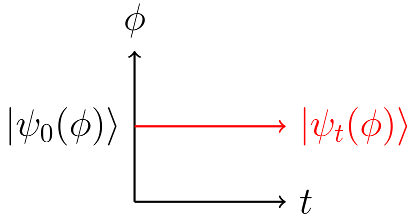


- Gapped matrices homotopic to:

$$U(n+m)/U(n) \otimes U(m)$$

3.16 Time dependent Feynman Hellman

- $i\partial_t |\psi\rangle = H |\psi\rangle$
- $\underbrace{\langle\psi_t| \partial_\phi H |\psi_t\rangle}_{\text{virtual work}} = i\, d_t\big(\underbrace{\langle\psi_t|\partial_\phi\psi_t\rangle}_{geometric}\big)$



- Proof:

$$\begin{aligned}
\langle\psi| \partial_\phi H |\psi\rangle &= \partial_\phi \langle\psi| H |\psi\rangle - \langle\partial_\phi\psi| H |\psi\rangle - \langle\psi| H |\partial_\phi\psi\rangle \\
&= \partial_\phi \langle\psi| \underbrace{H}_{id_t} |\psi\rangle - \langle\partial_\phi\psi| \underbrace{H}_{id_t} |\psi\rangle - \underbrace{\langle\psi| H}_{\langle H\psi|} |\partial_\phi\psi\rangle \\
&= i\partial_\phi\langle\psi|d_t\psi\rangle - i\langle\partial_\phi\psi|d_t\psi\rangle + i\langle d_t\psi|\partial_\phi\psi\rangle \\
&= d_t(i\langle\psi|\partial_\phi\psi\rangle)
\end{aligned}$$

4 Acknowledgment

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