Adiabatic Quantum Transport II

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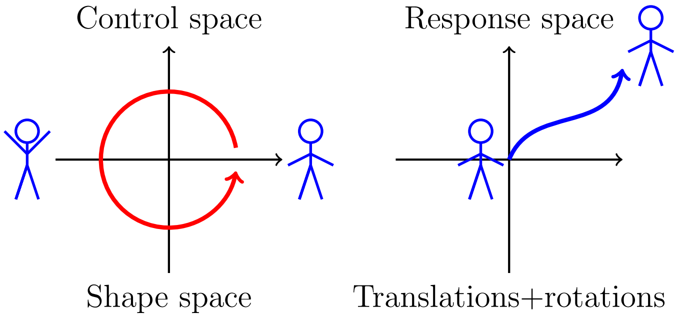
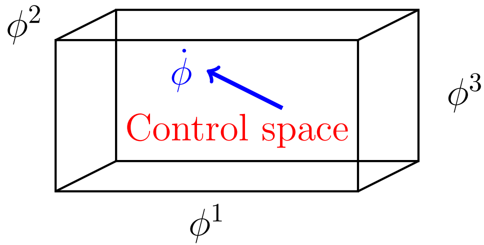
1 Controls and Response



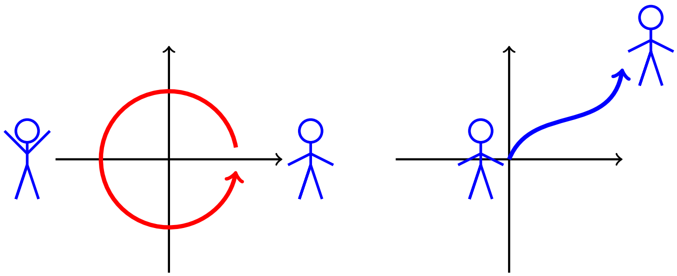
1.1 Driving and response



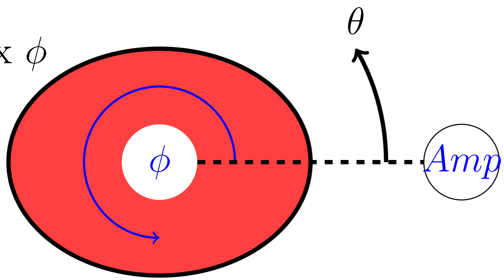
- Controls: ϕ
- Hamiltonian $H(\phi)$
- Driving: $\underbrace{\dot{\phi} = \frac{d\phi}{dt}}_{vector}$
- Response: $\underbrace{d(\text{virtual work})}_{1-form} = -dH = -\frac{\partial H}{\partial \phi}\, d\phi$
- Control spaces with interesting topology
- H is N-particle Hamiltonian.

1.2 Linear response

- Transport coefficients $(\partial_j H) = R_{jk}\dot{\phi}^k$
- Dissipation (power): $(\partial_j H)\dot{\phi}^j$
- Non-dissipative transport: $(\partial_j H)\dot{\phi}^j = 0 \iff R_{jk} = -R_{kj}$
- Total response: $\int(\partial_j H)dt = \underbrace{\mathbb{E}\left(R_{jk}\right)}_{\text{motion per stroke}} \delta\phi^k$
- Compute R form QM.



1.3 Driving loop current



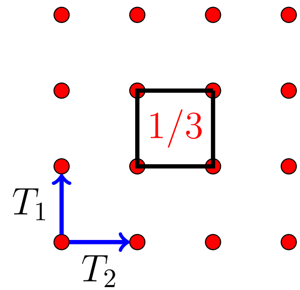
- Control: Aharonov-Bohm flux ϕ
- $\mathbf{A} = \frac{\phi}{2\pi} d\theta = \phi\, \delta(\theta) d\theta$
- Drive: emf $\dot{\phi} = -\oint E_j dx^j$
- Minimal coupling: $H(\mathbf{p} - \mathbf{A})$
- Heisenberg equation: $\mathbf{v} = i[H, \mathbf{x}] = \partial_p H$
- Response : Loop current

$$\begin{aligned} -dH &= -(\partial_\phi H)\, d\phi \\ &= \frac{1}{2}(\mathbf{v} \cdot \partial_\phi \mathbf{A} + \partial_\phi \mathbf{A} \cdot \mathbf{v})\, d\phi \\ &= \frac{1}{2}(v_\theta \delta(\theta) + \delta(\theta) v_\theta)\, d\phi \end{aligned}$$

- Choosing A is choosing a cross section.

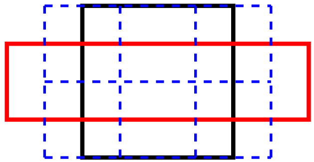
1.4 Electric field driving current

- Bloch Hamiltonians: Hofstadter model
- $H(\mathbf{k}) = e^{ik_1}\mathbf{T}_1 + e^{ik_2}\mathbf{T}_2 + h.c.$
- $\mathbf{T}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$
- $B = 1/3 \Leftrightarrow \omega = e^{2\pi i/3}$
- Control: Brillouin zone $\quad \mathbf{k} \in T^2$
- Driving: External electric field: $\dot{\mathbf{k}} = \mathbf{E}$
- Response: $-dH = -\underbrace{(\partial_k H)}_{velocity} dk = -i(T_j - T_j^*)dk^j$

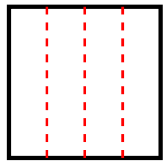


1.5 Controlling the metric

- Control: Metric tensor g
- Linear deformations L
- Active :$\mathbf{x}' = L^{-1}\mathbf{x}$; Passive $g' = L^t\, gL$
- $(d\ell)^2 = (d\mathbf{x}'^t)g'(d\mathbf{x}') = (d\mathbf{x}^t)\, g\, (d\mathbf{x})$
- Driving: strain rate $= \dot{g}$
- Response: Stress $dH(g) = -(\partial_g H)\, dg$



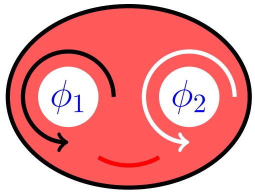
Active



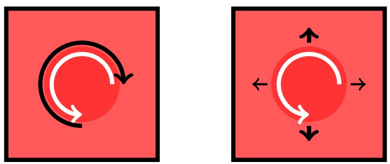
Passive

1.6 Odd (non-dissipative) response

- White arrow: Driving.
- Black arrow response.



Charge transport



Viscosity

1.7 Odd and Isotropic R in 2D

- Odd+isotropic 2-nd rank: $\propto \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{Levi-Civita } = i\sigma_y}$
- Viscosity: symmetric tensors $\mapsto$ symmetric tensors
- Basis for symmetric tensors in 2-D

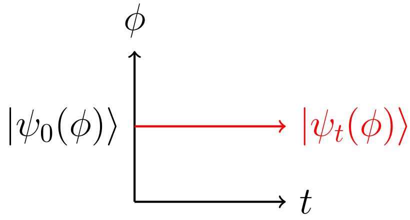
$$\boldsymbol{\sigma} = \underbrace{(\mathbb{1}, \sigma_x, \sigma_z)}_{\text{symmetric tensors}}$$

- Odd isotropic 4-th rank : $\propto \left(\sigma_1 \otimes \sigma_3 - \sigma_3 \otimes \sigma_1\right)$

Exercise 1. *Show that in 3 D: Isotropy + no dissipation implies vanishing transport.*

1.8 Time dependent Feynman Hellman

- $i\partial_t |\psi\rangle = H |\psi\rangle$
- $\underbrace{\langle\psi_t| \partial_\phi H |\psi_t\rangle}_{\text{virtual work}} = i\, d_t\big(\underbrace{\langle\psi_t|\partial_\phi\psi_t\rangle}_{geometric} \big)$



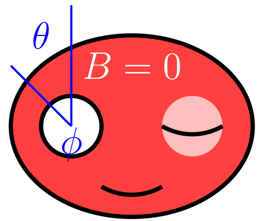
- Proof:

$$
\begin{aligned}
\langle\psi| \partial_\phi H |\psi\rangle &= \partial_\phi \langle\psi| H |\psi\rangle - \langle\partial_\phi\psi| H |\psi\rangle - \langle\psi| H |\partial_\phi\psi\rangle \\
&= \partial_\phi \langle\psi| \underbrace{H}_{id_t} |\psi\rangle - \langle\partial_\phi\psi| \underbrace{H}_{id_t} |\psi\rangle - \underbrace{\langle\psi| H}_{\langle H\psi|} |\partial_\phi\psi\rangle \\
&= i\partial_\phi\langle\psi|d_t\psi\rangle - i\langle\partial_\phi\psi|d_t\psi\rangle + i\langle d_t\psi|\partial_\phi\psi\rangle \\
&= d_t(i\langle\psi|\partial_\phi\psi\rangle)
\end{aligned}
$$

2 Topology of control space

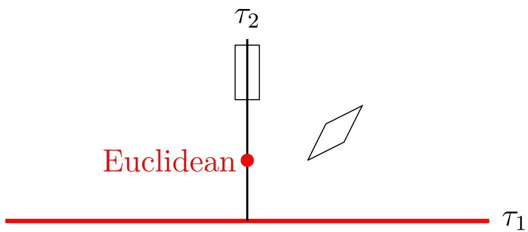
2.1 Flux torus

- Flat: $B = 0$ on red region
- Flux ϕ through hole: $\oint A = \phi$
- Aharonov-Bohm $H(\phi) = U H(\phi + 2\pi) U^*$
- Proof:
  - Take $U = e^{i\theta}$
  - $-iU^* dU = -d \log U$
  - $\oint A \mapsto \phi + 2\pi$
- ϕ is an angle: Flux space is a torus



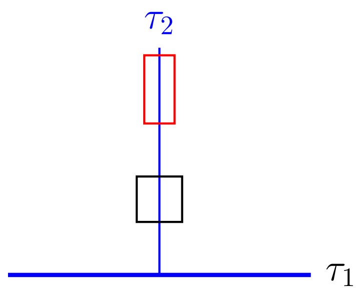
2.2 Flat metrics with $\det g = 1$

- $g(\tau) = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \quad \tau = \tau_1 + i\,\tau_2 \quad \tau_2 > 0$
- $\det g = \frac{|\tau|^2 - \tau_1^2}{\tau_2^2} = 1$
- $g(\tau)$ flat
  - curvature=0
  - No holonomy of parallel transport.



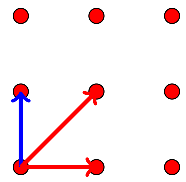
2.3 The metric on the space of metrics (Moduli)

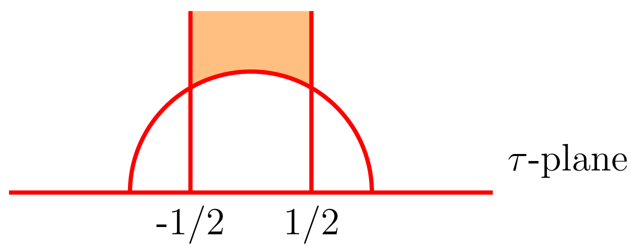
- $SL(2,\mathbb{R}) : L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det L = 1, \quad a,b,c,d \in \mathbb{R}$
- $g = (SL)^t \mathbb{1} (SL) = (SL)^t \cancel{SO(2)}^t \mathbb{1} \cancel{SO(2)} (SL)$
- $\tau \leftrightarrow SL(2,\mathbb{R})/SO(2)$
- $g'(\tau') = g(\tau), \quad g' = L^t\, g L, \quad \tau' = \frac{a\tau+b}{c\tau+d}$
- Haar measure on upper half-plane $\frac{d\tau_1 d\tau_2}{\tau_2^2}$



2.4 The fundamental domain of $SL(2,\mathbb{Z})$

- Torus: $\mathbb{R}^2/\mathbb{Z}^2$
- $L \in SL(2,\mathbb{Z}) : \mathbb{Z}^2 \mapsto \mathbb{Z}^2$
- $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$:
- Fundamental domain
  - 2D
  - Finite area
  - Two conic points
  - 1 cusp



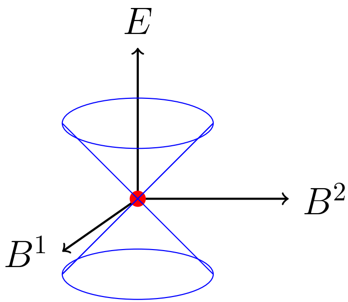


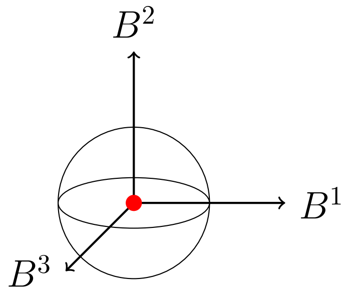
2.5 Gapped Hamiltonians

- Gap condition: Endows (Euclidean) control space with interesting topology

Example 2.1.

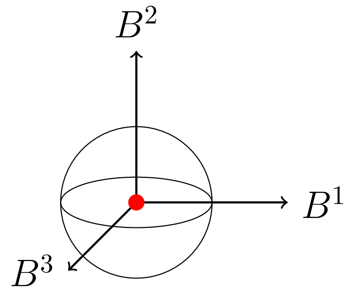
$$H(\mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\sigma}, \quad \mathbf{B} \in \underbrace{\mathbb{R}^3/0}_{S^2}$$





2.6 Wigner von Neuman

- Simple=Non-degenerate
- dim(Hermitian $n \times n$) $= n^2$
- dim(Hermitian, 2-fold degenerate) $= n^2 - 3$



2.7 Proof

- Simple: $H = U(n)\cancel{U(1)^{\otimes n}} \underbrace{D}_{diagonal} \cancel{U^*(1)^{\otimes n}} U^*(n)$

- $H \sim \underbrace{\text{cone}(\lambda_1 < \lambda_2 \cdots < \lambda_n)}_{eigenvalues} \otimes \underbrace{U(n)/U(1)^n}_{frame}$

- 1-eigenvalue crossing

$$H = U(n)\cancel{U(1)^{\otimes n-2}} \oplus \cancel{U(2)} \underbrace{(D_{n-2} \oplus \mathbb{1}_2)}_{diagonal \cancel{U^*(1)^{\otimes n-2}}} \oplus \cancel{U(2)} U^*(n)$$

$$H \sim \text{cone}(\lambda_1 = \lambda_2 \cdots < \lambda_n) \otimes U(n)/ \underbrace{U(2)}_{\mathbb{1}_2 V = V \mathbb{1}_2} \otimes U(1)^{n-2}$$

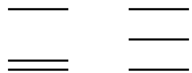
- Co-dimension: $\underbrace{1}_{cone} + \underbrace{4}_{U(2)} - \underbrace{2}_{U(1)} = 3$

Exercise 2. *co-dim (real symmetric)* $= 2$

2.8 Homotopy of simple matrices

- (simple hermitian $n \times n$) $\underbrace{\sim}_{homotopic}$ $U(n)/U(1)^n$

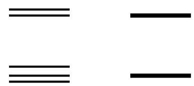
- Deform $(\lambda_1 < \lambda_2 \dots \lambda_n)$ to $(1, 2, \dots, n)$.



- (simple 2×2) $\sim S^2$

$$\underbrace{\pi_{0,1}(S^2) = 0, \quad \pi_{2,3}(S^2) = \mathbb{Z}, \quad \pi_{5,6}(S^2) = \mathbb{Z}_2}_{\text{rich topology}}$$

2.9 Homotopy of gapped matrices

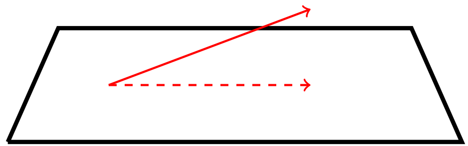


- Gapped matrices homotopic to:

$$U(n+m)/U(n)\otimes U(m)$$

3 Geometry of controlled projections

3.1 Projections: $P^2 = P$

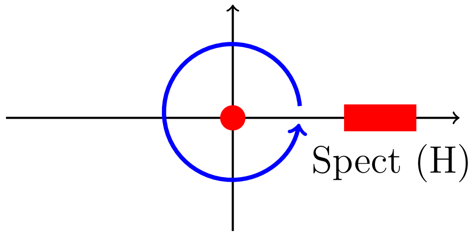


- Pure quantum state

$$P = |\psi\rangle \langle\psi| \,, \quad \mathrm{Rank} P = 1$$

- Spectral (aka Riesz) projection

$$P = -\frac{1}{2\pi i} \oint R(z)dz, \quad R(z) = \frac{1}{H - z}$$

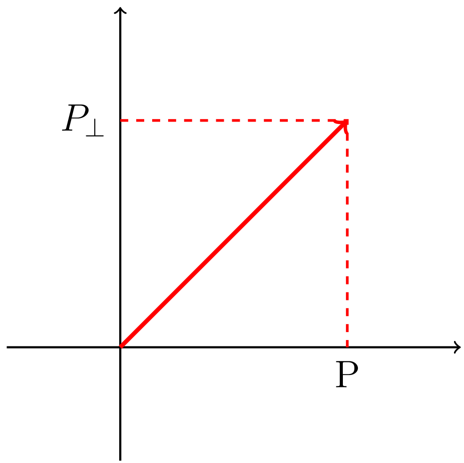


Exercise 3. *Use*

$$(z - z')R(z)R(z') = R(z') - R(z)$$

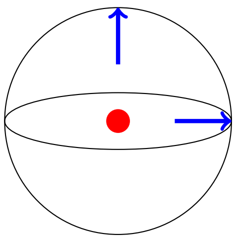
to show that the Riesz integral is a projection.

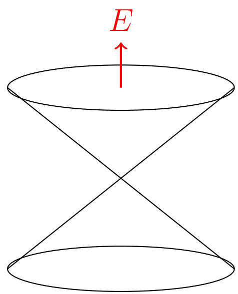
- Complementary projection:
- $P_\perp = \mathbb{1} - P$
- $P_\perp^2 = P_\perp, \quad PP_\perp = P_\perp P = 0$



3.2 Smooth Families of Projections

- $H(\mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\sigma}$
- Projection on ground state $P(\mathbf{B}) = \frac{\mathbb{1} - \hat{\mathbf{B}} \cdot \boldsymbol{\sigma}}{2}$
- $(\hat{\mathbf{B}} \cdot \boldsymbol{\sigma})^2 = \mathbb{1} \implies P^2 = P$
- Smooth away from $\mathbf{B} = 0$





3.3 dP

- Truly useful: $P\,(dP)\,P = 0$
- Proof:

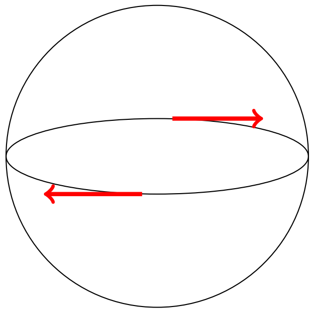
$$P^2 = P \Longrightarrow PdP + dPP = dP \Longrightarrow \cancel{P^2 dP} + PdPP = \cancel{PdP}$$

- Corollary:

$$PdP = P(dP)P_{\perp} = (dP)P_{\perp}$$

3.4 Parallel transport

- Parallel transport = No motion in Range P
- Parallel transport geometry surface
  - Preserves length
  - Preserves angle with geodesic



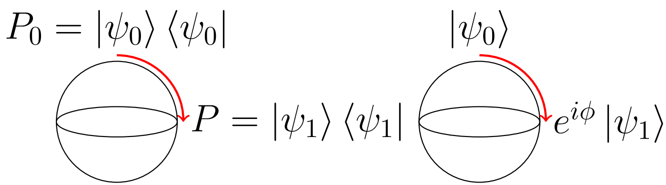
3.5 Motion of vectors in Range P

- $|\psi\rangle = P\,|\psi\rangle$
- $d\,|\psi\rangle = (dP)\,|\psi\rangle + Pd\,|\psi\rangle$
- $P_\perp d\,|\psi\rangle = (dP)\,|\psi\rangle$
- Need an equation for $Pd\,|\psi\rangle$
- Parallel transport is the choice: $\quad Pd\,|\psi\rangle = 0$

3.6 The "geodesic" equation

- The constraint to be in P: $P_{\perp} d\left|\psi\right\rangle = (dP)\left|\psi\right\rangle$
- Together with Parallel transport : $Pd\left|\psi\right\rangle = 0$
- Give the "Geodesic" equation:

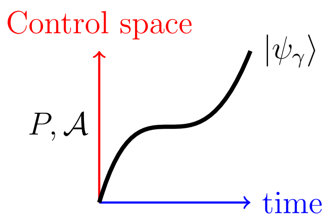
$$\begin{aligned} d\left|\psi\right\rangle &= (dP)\left|\psi\right\rangle \\ &= (dP)P\left|\psi\right\rangle \\ &= [dP, P]\left|\psi\right\rangle \end{aligned}$$



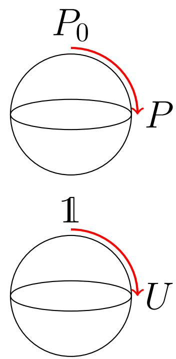
3.7 Connections for projections

$$\underbrace{\mathcal{A} = -i[dP, P]}_{Connection}, \quad \underbrace{D = d - i\mathcal{A}}_{\text{Covariant derivative}}, \quad \underbrace{D\,|\psi\rangle = 0}_{\text{Parallel transport}}$$

- $P, \mathcal{A}$: functions on control space.
- $|\psi_\gamma\rangle$, solution of ODE: function on space of paths γ.



3.8 Unitary intertwining evolutions



- $\underbrace{PU = UP_0}_{intertwining}$
- Generator $\mathcal{A} = i(dU)U^*$
- The equation for the generator

$$\underbrace{dP = i[\mathcal{A}, P]}_{\text{commutator equation}}$$

- Proof

$$\begin{aligned} P_0 = U^*PU \Longrightarrow 0 = & \quad (dU^*)PU + U^*(dP)U + U^*PdU \\ & = U\left((dU^*)PU + U^*(dP)U + U^*PdU\right)U^* \end{aligned}$$

$$0 = U(dU^*)P + (dP) + P(dU)U^* = i\mathcal{A}P + (dP) - iP\mathcal{A}$$

3.9 Solutions of the commutator equation

- If $\mathcal{A}$ solves $dP = i[\mathcal{A}, P]$
- Then $\mathcal{A} + \underbrace{\text{Commutant of } P}_{ambiguity}$ also a solution
- A special solution is Parallel transport

$$\mathcal{A} = -i[dP, P]$$

Proof.

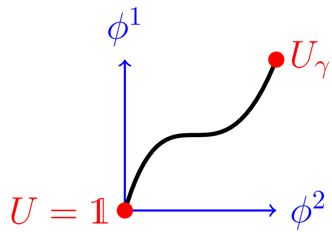
$$i[\mathcal{A}, P] = \big[[dP, P], P\big] = (dP)P - 2\cancel{P(dP)P} + PdP = dP$$

□

3.10 Evolution in spectral subspaces

- P a spectral projection for H.
- $\underbrace{P = P^*}_{orthogonal} \Longrightarrow \underbrace{\mathcal{A} = \mathcal{A}^*}_{hermitian} \Longrightarrow DU = 0, \quad U(0) = \mathbb{1}$
- Generates unitary intertwining evolution U_γ,
- Parallel transport is re-parametrization invariant

$$dU = -i\,[dP, P]\,U + \cancel{(\text{commutant } P)U}\,dt$$

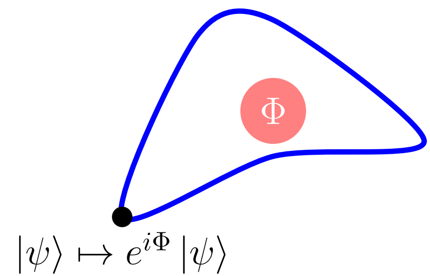


3.11 Holonomy

- Closed, anchored path

$$U_\gamma P(\phi_0) = P(\phi_0) U_\gamma$$

- AKA Berry's phase.
- Anchoring point matters (in non-abelian case).



3.12 2π rotation of a 2-level system

- 2-level system =Qubit
- Projections on equator: $P(\phi) = \frac{\mathbb{1}+\sigma_x \cos\phi + \sigma_y \sin\phi}{2}$
- Parallel transport= z-rotations $\underbrace{-i[dP, P] = \frac{\sigma_z}{2}}_{\text{generator of rotations}}$
- Intertwining: $U(\phi) = e^{i\sigma_z \phi/2} = \mathbb{1}\cos\phi/2 + i\sigma_z \sin\phi/2$
- Holonomy: $U(2\pi) = -\mathbb{1}$
- $\pi_1(SO(3)) = \mathbb{Z}_2$