Adiabatic Quantum Transport IV

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1 Intrim Summary

- $\underbrace{H(\phi^1,\phi^2)}_{\text{periodic matrix}} = H(\phi^1+2\pi,\phi^2) = H(\phi^1,\phi^2+2\pi)$

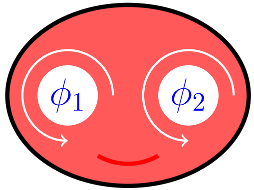
- Driving: $\dot{\phi}^j$

- Response': $\partial_k H$

- Smooth spectral projection $\underbrace{P(\phi^1,\phi^2)}_{rank=1}$

- Average transport coefficients quantized:

$$\underbrace{\mathbb{E}\left(\partial_2 H\right)}_{\text{average response}} = \underbrace{Chern(P|T^2)}_{\in\mathbb{Z}}\,\dot{\phi}^1$$



2 Fredholm index

- Positive: $F^*F \geq 0, \quad \langle\psi| F^*F |\psi\rangle = \langle F\psi|F\psi\rangle \geq 0$
- F Fredholm: $0 \in \text{spec}(F^*F)$ isolated eigenvalue of finite multiplicity.

Fredolm　　　　Non-Fredolm

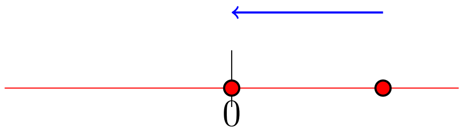
- $\text{Index}\, F = \dim\ker F^*F - \dim\ker FF^* \in \mathbb{Z}$

Exercise 1. *$F + \varepsilon B$ is Fredholm if B bounded and ε small.*

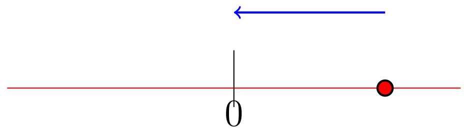
2.1 Source of stability

- Kernel is unstable, why is the index stable?
- spect $(F^* F)/0 = \text{spect}\,(F F^*)/0$
- proof
  - $F^* F\,|\psi\rangle = \mu\,|\psi\rangle\,, \quad F F^*\,|\phi\rangle = \mu\,|\phi\rangle\,, \quad F\,|\psi\rangle = |\phi\rangle$
  - Fail if

$$|\phi\rangle = F\,|\psi\rangle = 0 \Longrightarrow \mu = 0$$



Spec $(F F^*)$



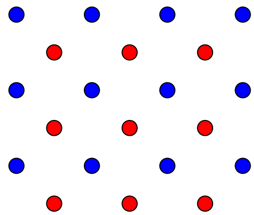
Spec $(F^* F)$

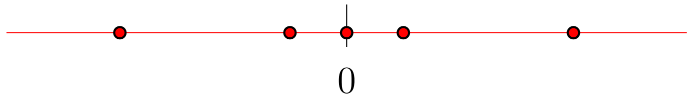
2.2 Rectangular matrices

- F an $n \times m$ matrix:
-

$$
\begin{aligned}
Index\, F &= \dim\ker F^*F - \dim\ker FF^* \\
&= \dim\ker \varepsilon F^*F - \dim\ker \varepsilon FF^* \\
&= Tr(\mathbb{1}_{n\times n} - \varepsilon F^*F) - Tr(\mathbb{1}_{m\times m} - \varepsilon FF^*) \\
&= n - m
\end{aligned}
$$

2.3 Zero modes



- Bi-partite graph: $|A|$, $|B|$
- $H = \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}, \quad \underbrace{A \leftrightarrow B}_{hop}$
- $H\sigma_z = -H\sigma_z$
- 

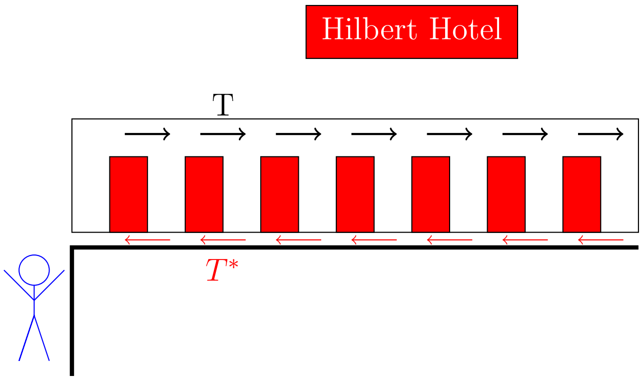
$$\begin{aligned} \#\text{zero modes} &= \dim\ker F^*F + \dim\ker FF^* \\ &\geq \Big| \dim\ker F^*F - \dim\ker FF^* \Big| \\ &= \big||A| - |B|\big| \end{aligned}$$

2.4 Hilbert hotel: The mother of Fredholm

T right shift on $\mathbb{Z}_+$

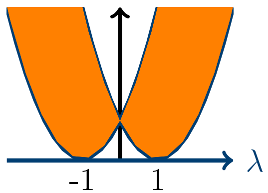
$$T^*T = \mathbb{1}, \quad TT^* = \mathbb{1} - |0\rangle\langle 0|$$

$$Index\, T = -1$$



2.5 How the index changes?

- $$Index\,(\mathbb{1} + \lambda T) = \begin{cases} 0 & |\lambda| < 1 \\ -1 & |\lambda| > 1 \end{cases}$$
- $F^*F = (1+\lambda^2)\mathbb{1} + \lambda(T+T^*)$
- $spect\,(T+T^*) = [-2,2], \quad spect(F^*F) = [(1-\lambda)^2, (1+\lambda)^2]$



2.6 Comparison of projections

- P, Q orthogonal, (possibly ∞ dimensional) projections.
- $(P-Q)^{2n+1}$ trace class
-

$$\underbrace{Tr(P-Q)}_{definition} = Tr(P-Q)^{2n+1} = \cdots = \dim\ker(P-Q-1) - \dim\ker(P-Q+$$

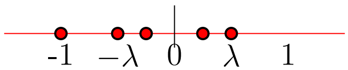
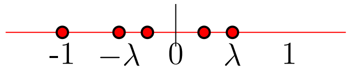


Figure 1: spectrum $P-Q$

2.7 Anti-commutative Pythagoras

- $C = P - Q, \quad S = P_\perp - Q$
- $C^2 + S^2 = \mathbb{1}, \quad CS + SC = 0$
- $\underbrace{(P - P_\perp)^2}_{(C-S)^2} = \mathbb{1}, \quad \underbrace{(Q_\perp - Q)^2}_{(C+S)^2} = \mathbb{1}$



Proof.

$$C\,|\psi\rangle = \lambda\,|\psi\rangle \Longrightarrow SC\,|\psi\rangle = \underbrace{\lambda(S\,|\psi\rangle) = -C(S\,|\psi\rangle)}_{anti-commutative}$$

Fail if

$$S\,|\psi\rangle = 0 \Longrightarrow \underbrace{C\,|\psi\rangle = \pm\,|\psi\rangle}_{Pythagoras}$$

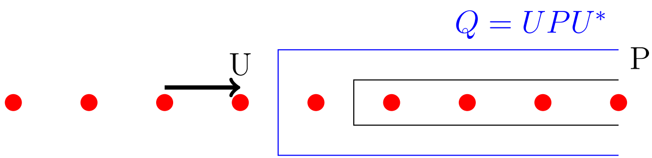
$\square$

2.8 Index of quasi-diagonal unitaries

- $Tr(P-Q)^3 = Tr\, PQ_\perp P - Tr\, QP_\perp Q \in \mathbb{Z}$
- $C^2 = P - PQ - QP + Q = PQ_\perp + QP_\perp$
- $C^3 = C^2 P - C^2 Q = PQ_\perp P - QP_\perp Q$
- $$U = \underbrace{\begin{pmatrix} * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * \end{pmatrix}}_{quasi-diagonal} = \begin{pmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--} \end{pmatrix}$$
- $$Tr(|U_{+-}|^2) - Tr(|U_{-+}|^2) = Tr(P \underbrace{U P_\perp U^*}_{Q_\perp} P) - Tr(\underbrace{U P U^*}_{Q} P_\perp \underbrace{U P U^*}_{Q}) \in \mathbb{Z}$$

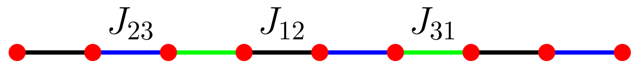
2.9 Unitarily related Projections

- $\dim P < \infty$ and $Q = UPU^*$ $\implies$ $Tr(P - Q) = 0$
- Not true if $\dim P = \infty$.
- $P\,|n\rangle = \theta(n)\,|n\rangle\,, \quad U\,|n\rangle = |n-1\rangle \Longrightarrow Tr(P - Q) = 1$
- Schrödinger vs Heisenberg



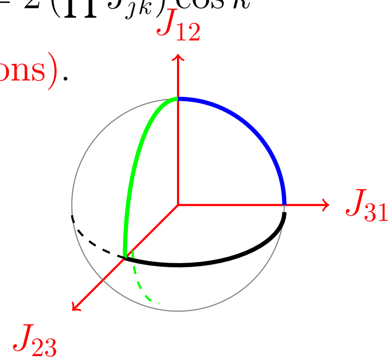
2.10 Thouless pump: Quantum superposition

- Bloch representation



$$H(J;k) = \begin{pmatrix} 0 & J_{12} & J_{31}e^{-ik} \\ J_{12} & 0 & J_{23} \\ J_{31}e^{ik} & J_{23} & 0 \end{pmatrix},$$

- Three energy bands
- $TrH = 0, \quad TrH^2 = \sum J_{jk}^2, \quad \det H = 2\left(\prod J_{jk}\right)\cos k$
- Pumping by interference (superpositions).
- $U = -(T^*)^2 \otimes P_0 + T \otimes (P_+ - P_-)$



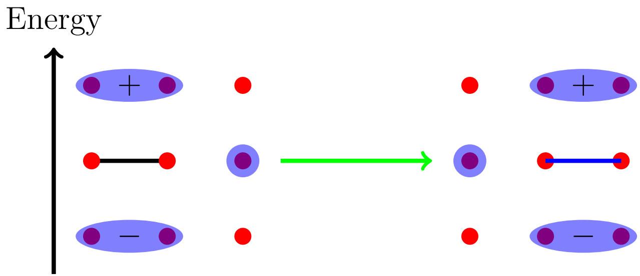
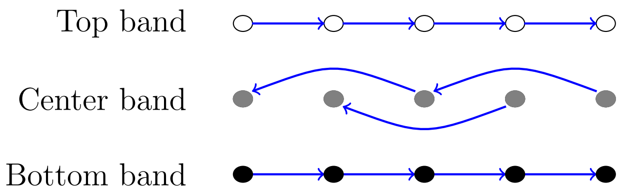
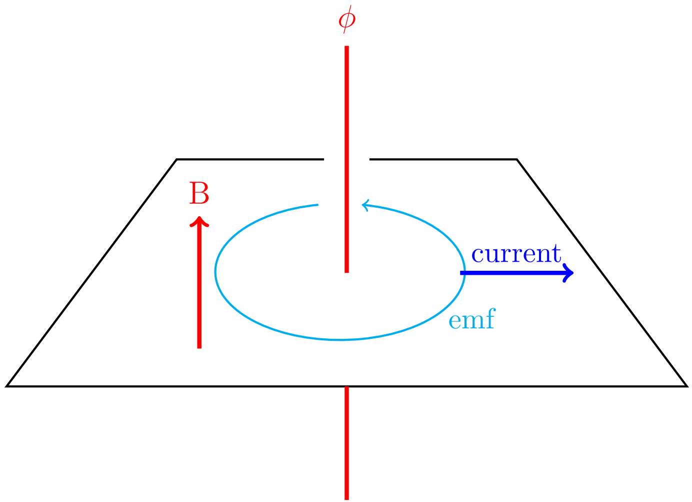


Figure 2: Particle motion generated by green great arc, in the three bands



2.11 Laughlin pump: Landau level



- Landau Hamiltonian driven by emf of point-like flux tube

$$H(\phi) = \frac{1}{2}\left(p - \frac{1}{2}B \times x - \phi A\right)^2, \quad A = \frac{\hat{\theta}}{r}$$

- Aharonov-Bohm period: $H(2\pi) = UH(0)U^*$

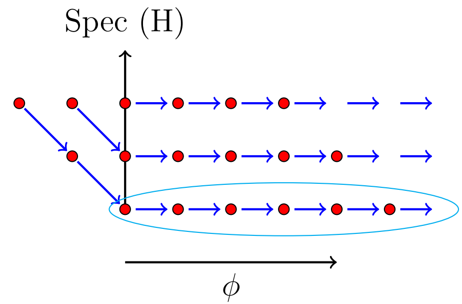


Figure 3: Laughlin pump: Spectral flow

2.12 Charge pumped=Index

- P: projection below a gap
- U the unitary of Qflux

$$U(\mathbf{x}) = \frac{x+iy}{|x+iy|}, \quad \mathbf{x} = (x,y)$$

$$Q = UPU^*$$

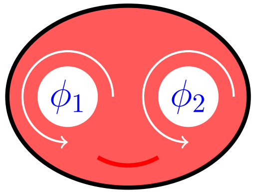
- Gap $\Longrightarrow$ P is quasi-diagonal

$$\langle \mathbf{x}| P |\mathbf{x}'\rangle = O\left(e^{-g|\mathbf{x}-\mathbf{x}'|}\right)$$

- $(P-Q)^2$ is trace class
- $Tr\,(P-Q)^3 \in \mathbb{Z}$

3 Open and closed

- Average transport= Chern (TKNN)
  - Too general
  - Averaging: Hastings Michalakis
- Transport=Fredholm index (Belissard)
  - Non-interacting
  - Fock space
  - Gap condition
- Open quantum systems
- FQHE beyond Laughlin
- Accuracy



4 Acknowledgment

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