Dynamical phase transition in random lattice triangulations.

Pietro Caputo,
with Fabio Martinelli, Alistair Sinclair, and Alexandre Stauffer

La Sapienza, Roma
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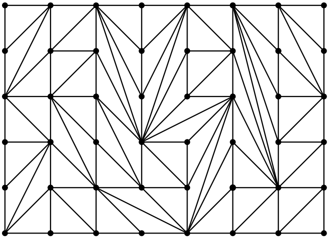
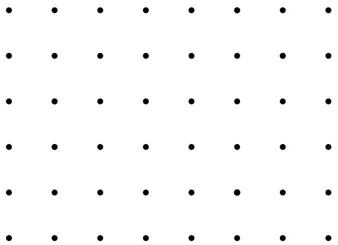
Plan



- Lattice triangulations: basic facts
- How to simulate a random LT ? *Flip dynamics*
- Weighted random triangulations: Phase transition
- Mixing time estimates

Lattice triangulations: basic facts

Triangulation of a $m \times n$ rectangle $R_{m,n}$ in $\mathbb{Z}^2$:



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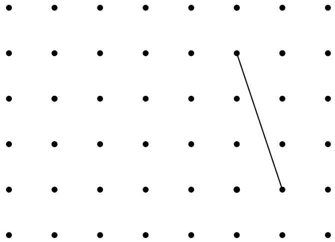

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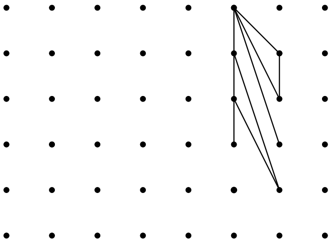

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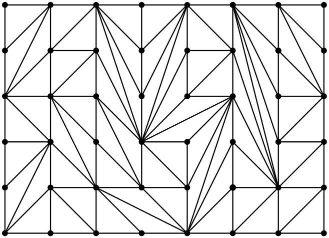
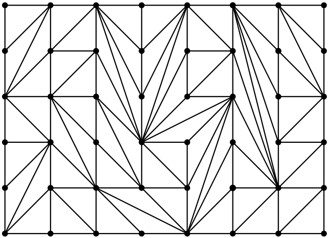


Maximal set of non-intersecting edges. Edges are segments joining two lattice points, containing no other lattice point.

Lattice triangulations: basic facts

Call $\Omega(m, n)$ the set of all triangulations of $R_{m,n}$



Any $\sigma \in \Omega(m, n)$ has:

- $2mn$ triangles, each with area $\frac{1}{2}$
- $3mn + m + n$ edges, with *fixed* midpoints

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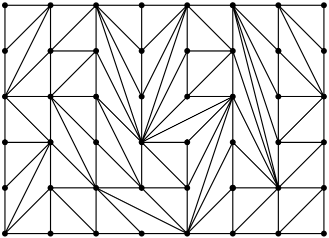

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Thus $\sigma = \{\sigma_x\,,\ x \in \Lambda_{m,n}\}$, where

$$\Lambda_{m,n} = \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots, m - \tfrac{1}{2}, m\} \times \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots, n - \tfrac{1}{2}, n\} \setminus R_{m,n}$$

Counting lattice triangulations

$\#\Omega(1, n) = \binom{2n}{n}$ Equivalence with lattice paths

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In general it is known that

$$(4.15)^{mn} < \#\Omega(m, n) < (6.86)^{mn}.$$

[Gel'fand-Kapranov-Zelevinski 1994, Yu-Orevcov 1999,
Anclin 2003, Kaibel-Ziegler 2003, Matousek-Valtr-Weltzl 2006]

Sampling lattice triangulations

Flip moves: an edge is flippable if it is the diagonal of a parallelogram.

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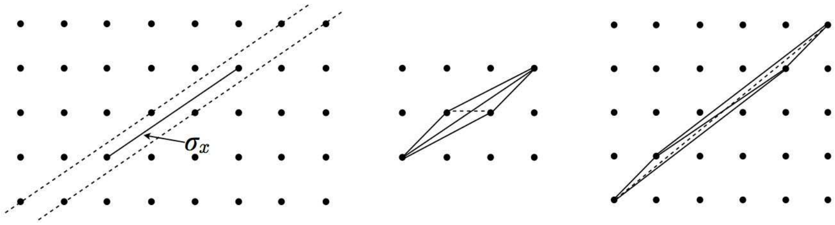

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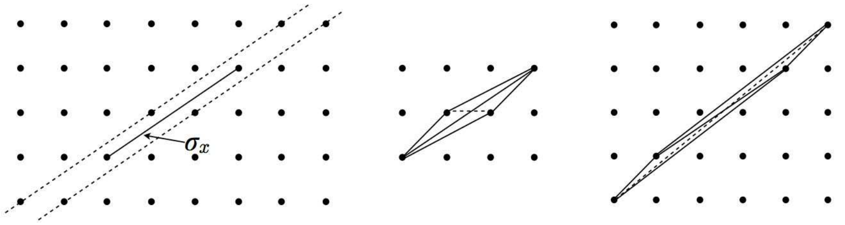

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The largest edge in a triangulation is always flippable to a shorter edge. The flip graph on $\Omega(m, n)$ is connected.

Markov chain: pick a midpoint at random, if flippable toss a coin to update it. This converges to the uniform distribution on $\Omega(m, n)$.

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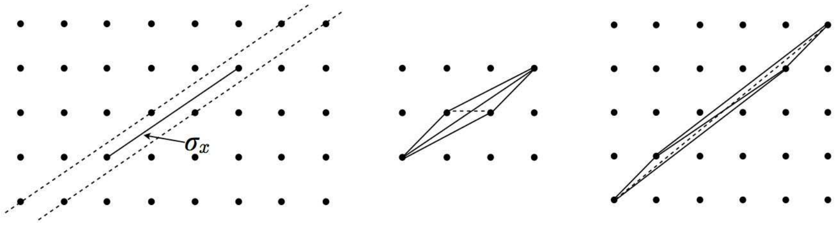

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Markov chain: pick a midpoint at random, if flippable toss a coin to update it. This converges to the uniform distribution on $\Omega(m, n)$.

PROBLEM: Is the Mixing Time $poly(n, m)$?

Weighted triangulations and Gibbs sampler

Consider the Gibbs distribution on $\Omega = \Omega(m, n)$: for $\lambda > 0$

$$\mu(\sigma) = \frac{\lambda^{|\sigma|}}{Z}, \qquad |\sigma| = \sum_{x \in \Lambda_{m,n}} |\sigma_x|$$

where $|\sigma_x|$ denotes the ℓ_1 length of σ_x.

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Gibbs sampler: pick a midpoint $x \in \Lambda_{m,n}$ at random, if the edge σ_x is flippable to edge σ'_x (producing a new triangulation σ'), then flip it with probability

$$\frac{\mu(\sigma')}{\mu(\sigma') + \mu(\sigma)} = \frac{\lambda^{|\sigma'_x|}}{\lambda^{|\sigma'_x|} + \lambda^{|\sigma_x|}}.$$

Repeating at each time unit independently yields a Markov chain with transition matrix P that is reversible w.r.t. μ:

$$\mu(\sigma)P(\sigma,\tau) = \mu(\tau)P(\tau,\sigma), \qquad \sigma, \tau \in \Omega.$$

Equivalently, $Pf(\sigma) = \sum_{\tau} P(\sigma,\tau)f(\tau)$ is self-adjoint in $L^2(\Omega, \mu)$.

Convergence to equilibrium

The Spectral Gap is defined by $\mathrm{gap} = 1 - \lambda_2(P)$, where $\lambda_2(P)$ is the second largest eigenvalue of P.
The Relaxation Time $T_{\mathrm{REL}} = \mathrm{gap}^{-1}$ is the rate of exponential decay to equilibrium in $L^2(\Omega, \mu)$:

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The Mixing Time is defined by

$$T_{\mathrm{mix}} = \inf \left\{ t \in \mathbb{N} \ : \ \max_{\sigma \in \Omega} \| P^t(\sigma, \cdot) - \mu \|_{\mathrm{TV}} \ \leqslant \ 1/4 \right\},$$

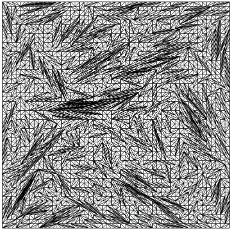
$P^t(\sigma, \cdot)$ is the distribution after t steps when the initial state is σ.
Total variation distance $\|\nu - \mu\|_{\mathrm{TV}} = \frac{1}{2} \sum_{\tau \in \Omega} |\nu(\tau) - \mu(\tau)|$.

The mixing time is the rate of exponential decay w.r.t. $\| \cdot \|_{\mathrm{TV}}$.

Standard: $T_{\mathrm{mix}} \ \leqslant \ Const. \times \ T_{\mathrm{REL}} \log(1/\mu_{\mathrm{min}})$.

Weighted triangulations and Gibbs sampler

Simulations suggest a phase transition (here $n = m = 50$):



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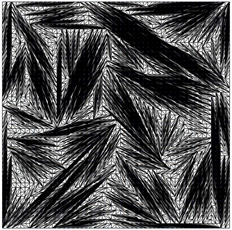

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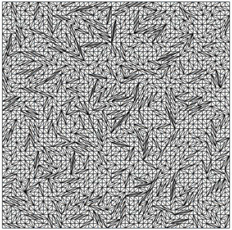

$\lambda = 0.9$

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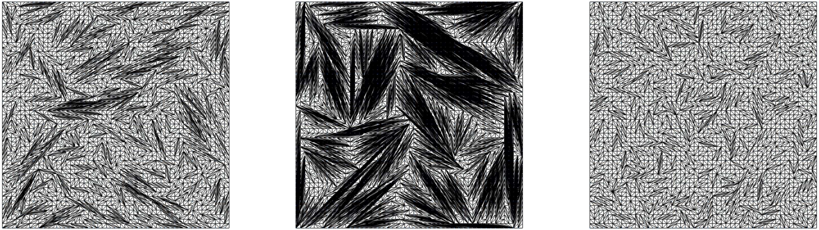

$\lambda = 1$ $\lambda = 1.1$ $\lambda = 0.9$

Conjectures

- $\lambda < 1$: $T_{\mathrm{mix}} = O(mn(n+m))$
- $\lambda > 1$: $T_{\mathrm{mix}} = \exp(\Omega(mn(n+m)))$
- $\lambda = 1$: $T_{\mathrm{mix}} = poly(m,n)$ (?)

Main results

Theorem (Rapid mixing for small λ)

There exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ one has $T_{\mathrm{mix}} = O(mn(m+n))$.

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Theorem (Slow mixing for $\lambda > 1$)

For all $\lambda > 1$ *one has* $T_{\text{mix}} \geqslant \exp(c(m+n))$.

[Sharper results for thin rectangles. More on this later]

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Theorem (Rapid mixing for small λ)

There exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ one has $T_{\mathrm{mix}} = O(mn(m+n))$.

Theorem (Slow mixing for $\lambda > 1$)

For all $\lambda > 1$ one has $T_{\mathrm{mix}} \geqslant \exp(c(m+n))$.

[Sharper results for thin rectangles. More on this later]

Theorem (Small λ phase: equilibrium results)

There exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ one has:

- *exponential tails for the marginals on $|\sigma_x|$,*
- *"exponential decay of correlations".*

Sketch of proof of rapid mixing for small λ

Path coupling (Bubley-Dyer 1997).

Exponential metric: Fix $\alpha > 1$, and for $\sigma, \tau \in \Omega(m, n)$ differing only at $x \in \Lambda_{m,n}$ set

$$\Delta(\sigma, \tau) = \left|\alpha^{|\sigma_x|} - \alpha^{|\tau_x|}\right|. \tag{1}$$

If $|\sigma_x| = |\tau_x| = 2$ (unit diagonals), then set $\Delta(\sigma, \tau) = \alpha^2 - 1$. Extend Δ to all pairs of triangulations by defining $\Delta(\sigma, \tau)$ to be the shortest path distance between σ and τ in the flip graph, with edge lengths given by the above weight.

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Lemma

For $\lambda < \lambda_0 = 1/8$, $\alpha = 1/\lambda_0$, there is a coupling such that

$$\mathbb{E}_{\sigma,\tau}[\Delta(\sigma',\tau')] \leqslant \Delta(\sigma,\tau)\left(1 - \tfrac{1}{2|\Lambda_{m,n}|}\right),$$

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From here it is standard to conclude $T_{\text{mix}} = O(|\Lambda_{m,n}|(m + n))$ (use Markov property, Markov's inequality, and the fact that $\max_{\sigma,\tau} \Delta(\sigma, \tau) \leqslant \exp c(m + n)$).

Sketch of proof of slow mixing for $\lambda > 1$

Bottleneck: If we find a set $A \subset \Omega(m, n)$ such that $\mu(A) \leqslant 1/2$ and

$$\frac{\mu(\partial A)}{\mu(A)} \leqslant e^{-c(m+n)},$$

then T_{mix} and T_{REL} are exponential in $m + n$. Here ∂A is the set of $\sigma \in A$ such that for some $x \in \Lambda_{m,n}$ a flip at x produces $\sigma' \notin A$.

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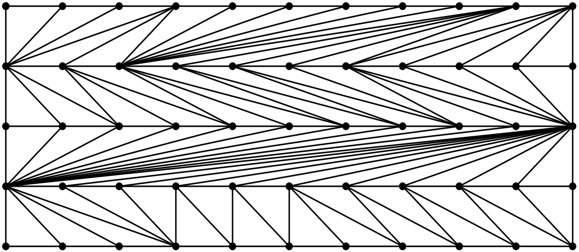
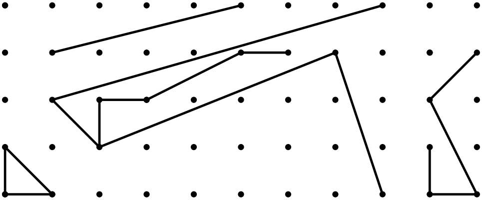
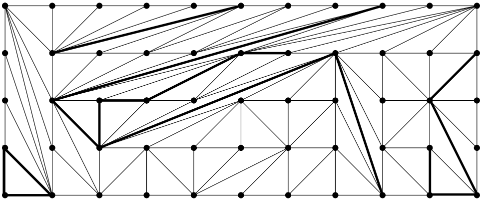
Idea: Pick A as the set of all herringbone triangulations:



Each layer has opposite orientation. Note that $\sigma \in \partial A$ iff an internal edge is vertical, which is exponentially unlikely given A.

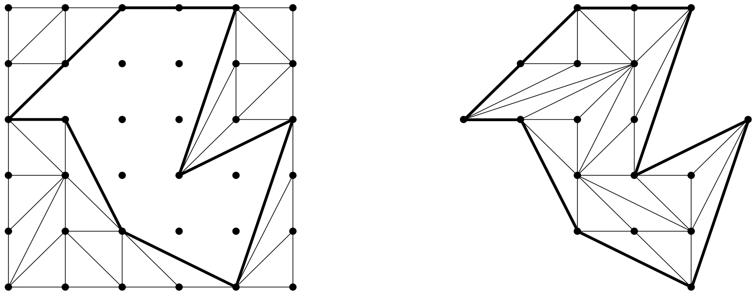
Lattice triangulations with constraints





Lattice triangulations with constraints

Triangulations of an arbitrary lattice polygon



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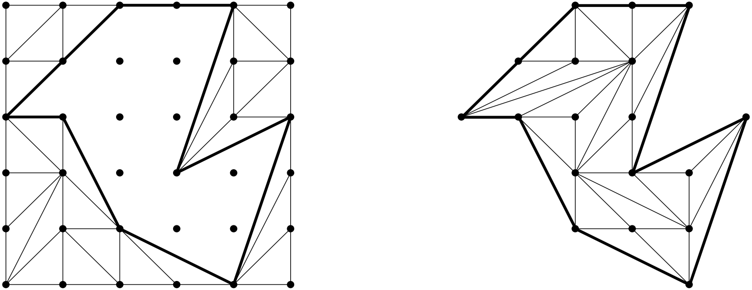

Lemma

For any set of constraints, the flip graph on triangulations is connected. Moreover, its diameter is $O(mn(m+n))$.

Remark: Rapid mixing results for $\lambda < \lambda_0$ hold uniformly in the set of constraints.

Ground states

A ground state is a triangulation $\bar{\sigma}$ with minimal total length. If there are no constraints, GS is trivial: all edges are unit vertical/horizontal or unit diagonal (GS is unique up to unit diagonal flips).

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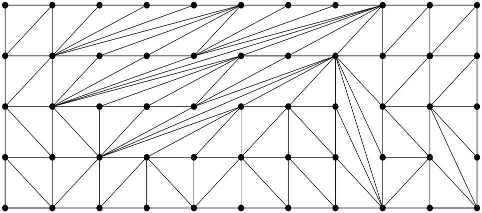
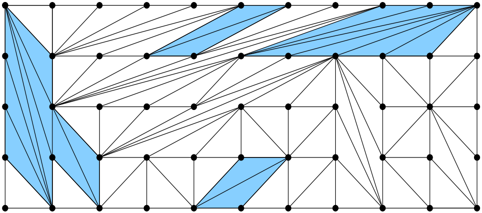
Lemma (Ground State Lemma)

Given any set of constraints, the ground state triangulation is unique (up to possible unit diagonal flips), and can be constructed by placing each edge in its minimal length configuration consistent with the constraints, independent of the other edges.



Peierls' type estimates for small λ

Think of the GS triangulation $\bar{\sigma}$ as a fixed background graph.
Call $\mathcal{S} = \mathcal{S}(\sigma)$ the (random) set of triangles that are not GS in σ.



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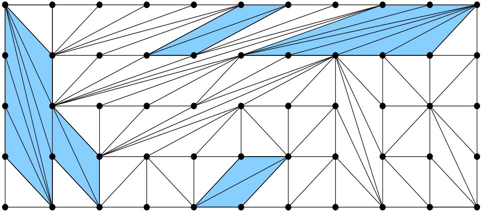

Lemma

There exists $\lambda_0 \in (0,1)$ such that uniformly in the constraints, for any fixed ground state region V,

$$\mu(\mathcal{S} \supseteq V) \leqslant (\lambda/\lambda_0)^{|V|/2}.$$

Exponential decay for small λ

For some $\lambda_0 \in (0,1)$, uniformly in the constraints, one has:

Theorem (Exponential tails)

For every point $x \in \Lambda$ and every $k \in \mathbb{N}$,

$$\mu(|\sigma_x| = |\bar{\sigma}_x| + k) \leqslant (\lambda/\lambda_0)^k.$$

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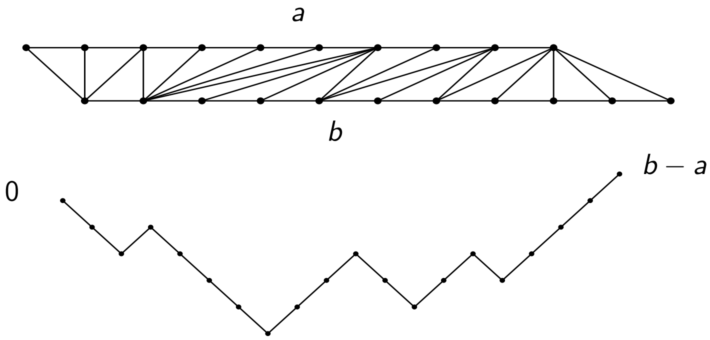
Influence of an extra constraint σ_x at $x \in \Lambda_{m,n}$ (a boundary condition). For $U \subset \Lambda_{m,n}$ call $d(U, \sigma_x)$ the "distance" in $\bar{\sigma}$ between σ_x and U.

Theorem (Spatial mixing)

For all $x \in \Lambda$, for all fixed σ_x, and $U \subseteq \Lambda$, one has

$$\|\mu^{\sigma_x} - \mu\|_{\mathrm{TV},U} \leqslant |U|(\lambda/\lambda_0)^{d(U,\sigma_x)}.$$

Optimal bounds on T_{mix} for $m = 1$



Theorem

- $\lambda < 1$: $T_{\mathrm{mix}} = \Theta(n^2)$ *(path coupling)*
- $\lambda > 1$: $T_{\mathrm{mix}} = \exp(\Omega(n^2))$ *(1 layer bottleneck)*
- $\lambda = 1$: $T_{\mathrm{mix}} \sim n^3 \log n$ *(coupling, e.g. D.B. Wilson 2001)*

Optimal bounds for thin rectangles: *m* fixed and *n* large

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Lower bound for $\lambda > 1$: improved version of herringbone bottleneck.

Upper bound for $\lambda < 1$: requires new ideas combined with results from A. Stauffer (2015) on the existence of a Lyapunov function for the general $m \times n$ triangulations.

Upper bound: $T_{\mathrm{mix}} = O(n^2)$

Burn-in phase: for some $T = c(\lambda)n^2$, uniformly in the initial condition, one has w.h.p.

$$\sigma(t) \in \widetilde{\Omega}\,, \qquad t \in [T, T + n^{10}],$$

$\widetilde{\Omega}$ is the set of triangulations such that $|\sigma_x| \leqslant C \log n$ for all x.

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Based on Lyapunov function: $\alpha > 1$, $\Psi(\sigma) = \sum_{x\in\Lambda_{m,n}} \alpha^{|\sigma_x|}$

Contraction: for any $\lambda < 1$, $\exists \alpha > 1$ such that

$$\mathbb{E}_\sigma[\Psi(\sigma')] \leqslant \left(1 - \tfrac{\varepsilon}{n}\right)\Psi(\sigma)\,, \qquad \sigma \notin \widetilde{\Omega}$$

[A. Stauffer (2015)]

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Note: it is now sufficient to prove $\widetilde{T}_{\mathrm{mix}} = O(n^2)$,
for the chain restricted to $\widetilde{\Omega}$.

Proof of $\widetilde{T}_{\mathrm{mix}} = O(n^2)$

Log-Sobolev inequality for chain restricted to $\widetilde{\Omega}$.

$$\alpha(n) := \inf_f \frac{\mathcal{E}(f)}{\mathrm{Ent}(f^2)}\,, \quad \mathrm{Ent}(f^2) = \mu[f^2 \log(f^2/\mu[f^2])].$$

$\mathcal{E}(f) = \frac{1}{2n} \sum_{\sigma,\sigma'} \mu(\sigma) P(\sigma,\sigma')(f(\sigma) - f(\sigma'))^2$.

Standard: $\widetilde{T}_{\mathrm{mix}} \leqslant C\,\alpha(n)^{-1} \log\log(1/\mu_*) \leqslant C\,\alpha(n)^{-1} \log n$.

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Theorem

$$\alpha(n) \;\geqslant\; n^{-1+o(1)}.$$

In particular, $\widetilde{T}_{\mathrm{mix}} = O(n^{1+o(1)}) = O(n^2)$.

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Lemma

$$\alpha(n) \geqslant n^{-C}.$$

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Main idea: Improved canonicals path argument.

Classical:
$T_{\mathrm{REL}} \leqslant \mathcal{C} := \max_{\eta \sim \eta'} \sum_{\sigma,\sigma' \colon (\eta,\eta') \in \Gamma_{\sigma,\sigma'}} \frac{\mu(\sigma)\mu(\sigma')}{\mu(\eta)p(\eta,\eta')} |\Gamma_{\sigma,\sigma'}|$

Here $\mathcal{C} \sim \exp(C\, n)$. We want polynomial.

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Idea: define reduced space $\Omega_0 \subset \widetilde{\Omega}$ such that on Ω_0 canonical paths have polynomial congestion ratio $\mathcal{C}_{\Omega_0} \leqslant n^C$.

If T_0 is time needed to enter Ω_0 with probab. $a > 0$, then

$$T_{\mathrm{REL}} \leqslant \frac{6\, T_0^2}{a} + \frac{3\, \mathcal{C}_{\Omega_0}}{a^2}.$$

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Also, we prove that uniformly in initial condition, the process enters Ω_0 within time $T_0 := c_1 n^2$ with probab. at least $a \geqslant 1/2$. This implies $\alpha(n) \geqslant n^{-C}$ for some $C > 0$.

Proof of $\alpha(n) \geqslant n^{-1+o(1)}$: Step 2

Recursive analysis proves

Lemma
$\alpha(n) \geqslant c\, n^{-1+o(1)}\, \alpha(\log^6 n)$.

Proof uses quasi-factorization of entropy under approximate bisection:

$$\mathrm{Ent}_\Lambda(f^2) \leqslant (1+n^{-\varepsilon})\mu\left[\mathrm{Ent}_{\Lambda_1}(f^2|\Lambda\setminus\Lambda_1)+\mathrm{Ent}_{\Lambda_2}(f^2|\Lambda\setminus\Lambda_2)\right].$$

$\Lambda=\Lambda_1\cup\Lambda_2$, $\Lambda_1\cap\Lambda_2$ is a rectangle $C\log n\times m$.

This requires exponential decay of correlations in thin rectangles for all $\lambda<1$.